# Short Papers

## Analyzing the Stability of the FDTD Technique by Combining the von Neumann Method with the Routh–Hurwitz Criterion

José A. Pereda, Luis A. Vielva, Ángel Vegas, and Andrés Prieto

*Abstract*—This paper addresses the problem of stability analysis of finite-difference time-domain (FDTD) approximations for Maxwell's equations. The combination of the von Neumann method with the Routh–Hurwitz criterion is proposed as an algebraic procedure for obtaining analytical closed-form stability expressions. This technique is applied to the problem of determining the stability conditions of an extension of the FDTD method to incorporate dispersive media previously reported in the literature. Both Debye and Lorentz dispersive media are considered. It is shown that, for the former case, the stability limit of the conventional FDTD method is preserved. However, for the latter case, a more restrictive stability limit is obtained. To overcome this drawback, a new scheme is presented, which allows the stability limit of the conventional FDTD method to be maintained.

*Index Terms*—Dispersive media, FDTD methods, Routh's methods, stability analysis, von Neumann method.

## I. INTRODUCTION

The finite-difference time-domain (FDTD) method is a powerful numerical technique for the solution of electromagnetic problems. Based on the finite-difference approximation of the time-dependent Maxwell's curl equations, this method leads to a conditionally stable scheme.

The stability of a difference scheme refers to the unstable growth or stable decay of errors in the arithmetic operations required to solve the finite-difference equations. Due to the conditional stability of the FDTD method, it is essential to choose a number of parameters—time step, size of the spatial mesh, etc.—in such a way that the difference scheme remains stable. Thus, the derivation of closed-form analytical stability conditions is of great theoretical and practical interest.

The establishment of the analytical stability condition for the conventional FDTD method (Yee's scheme for isotropic, nondispersive lossless media) was an early development [1]. More recently, stability conditions were determined for extensions of the conventional FDTD method to incorporate dispersive media [2], [3], anisotropic media, and lossy media [5], [6].

The von Neumann method is probably the most popular procedure for determining the stability of finite-difference approximations for space–time problems [7]. For a given difference scheme, this technique provides an associated polynomial. The condition for stability is that all the roots of this polynomial have modulus less than or equal to unity. When applying this method, the root locus is obtained as a function of the parameters of interest. This technique usually requires exhaustive numerical root searching, which makes the derivation of closed-form stability conditions difficult. The numerical searching may also intro-

duce inaccuracies in the location of the roots and, therefore, in the ranges of stability for the parameters of interest.

The Routh–Hurwitz (R–H) criterion is a procedure widely used to study the stability of continuous- and discrete-time linear systems [8]. This method is able to establish the location of the zeros of a real-coefficient polynomial with respect to the imaginary axis, without actually solving for the zeros.

This paper proposes combining the von Neumann method with the R–H criterion as an algebraic procedure for deriving closed-form stability expressions for the FDTD method. To illustrate this technique, it is applied to the stability analysis of an extension of the FDTD method, previously published, which is able to incorporate dispersive media [9]. It is shown that the scheme introduced in [9] to treat Debye media preserves the stability limit of the conventional FDTD method. However, a more restrictive stability limit is obtained for the scheme given in [9] to treat Lorentz media. To overcome this limitation, a new scheme is presented, which recovers the stability limit of the conventional FDTD method.

## II. THEORY

### A. von Neumann Method

Each difference scheme has a theoretical exact solution. However, when explicit calculations are carried out in a computer, errors are committed due to the finite precision of the arithmetic operations. The study of the stability of a finite-difference scheme consists of finding the conditions under which the error—difference between the theoretical and numerical solutions of the finite-difference equation—remains bounded as the number of time iterations tends to infinity.

The von Neumann method basically consists of considering a Fourier series expansion of the error at the mesh nodes at a given time instant $t = n\Delta_t$. Due to linearity, only a single term of this expansion needs to be considered, i.e.,

$$\mathcal{E}^n(i, j, k) = \mathcal{E}_0 Z^n e^{\mathbf{j}(i\Delta_x \tilde{k}_x + j\Delta_y \tilde{k}_y + k\Delta_z \tilde{k}_z)} \tag{1}$$

where $\mathcal{E}_0$ is a complex amplitude, indexes $i, j, k$ denote the position of the nodes in the mesh, $\Delta_\alpha (\alpha = x, y, z)$ are the sizes of the discretization cell, and $\tilde{k}_\alpha (\alpha = x, y, z)$ are the wavenumbers of the discrete modes in the $\alpha$-direction.

In (1), $Z$ is a complex variable, often called the amplification factor, which gives the growth of the error in a time iteration, i.e., $\mathcal{E}^{n+1}(i, j, k) = Z\mathcal{E}^n(i, j, k)$. To ensure that a finite-difference scheme will be stable, the error must not grow as the time increases, thus, the condition $|Z| \leq 1$ must be satisfied.

To derive the stability condition for a particular scheme, solutions of the form (1) are substituted into the difference equations. This leads to a characteristic polynomial in $Z$

$$S(Z) = \sum_{i=0}^{N} a_i Z^{N-i}. \tag{2}$$

The condition for stability can now be written as

$$|Z_i| \leq 1$$

where $Z_i$ are the roots of $S(Z)$. That is, the roots of $S(Z)$ must lie inside or on the unit circle in the $Z$-plane. To determine the range of the parameters of interest for which the scheme is stable, it is usually

necessary to perform exhaustive numerical searching for the roots of $S(Z)$, as a function of these parameters.

### B. R–H Criterion

This criterion establishes that the polynomial

$$S(r) = \sum_{k=0}^{N} b_k r^{N-k}, \qquad b_0 > 0$$

with constant real coefficients $b_k$ has no roots in the right-half of the $r$-plane if all the entries of the first column of the Routh table are nonnegative quantities. In order to build up the Routh table, the coefficients $b_k$ are arranged in two rows: the first row consists of the coefficients that correspond to even powers of $r$, and the second row consists of the coefficients that correspond to odd powers of $r$ (for simplicity in the notation, and without loss of generality, we assume $N$ to be an even number):

$$c_{1,k} = b_{2k}, \quad k = 0, 1, \ldots, N/2$$
$$c_{2,k} = b_{2k+1}, \quad k = 0, 1, \ldots, N/2 - 1.$$

The remaining entries of the table are obtained by using the following expression:

$$c_{j,k} = \frac{-1}{c_{j-1,1}} \begin{vmatrix} c_{j-2,1} & c_{j-2,k+1} \\ c_{j-1,1} & c_{j-1,k+1} \end{vmatrix}$$

where $j = 3, 4, \ldots, N$, and $k = 0, 1, \ldots, N/2$. Therefore, the Routh table is given by

| | | | | |
|---|---|---|---|---|
| $c_{1,0} = b_0$ | $\cdots$ | $c_{1,k} = b_{2k}$ | $\cdots$ | $c_{1,N/2} = b_N$ |
| $c_{2,0} = b_1$ | $\cdots$ | $c_{2,k} = b_{2k+1}$ | $\cdots$ | $c_{2,N/2} = 0$ |
| $c_{3,0}$ | $\cdots$ | $c_{3,k}$ | $\cdots$ | $0$ |
| $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $0$ |
| $c_{j,0}$ | $\cdots$ | $c_{j,k}$ | $\cdots$ | $0$ |
| $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $c_{N,0}$ | $0$ | $0$ | $0$ | $0$ |

The R–H criterion can be used to determine if any root of the stability polynomial $S(Z)$ lies outside the unit circle in the $Z$-plane. To this end, the bilinear transformation

$$Z = \frac{r+1}{r-1} \tag{3}$$

is applied to $S(Z)$. As a result, a polynomial $S(r)$ in the $r$-plane is obtained. The above transformation maps the exterior of the unit circle in the $Z$-plane onto the right-half of the $r$-plane. Therefore, if the polynomial $S(r)$ obtained by applying (3) to (2) has no roots in the right-half of the $r$-plane, the polynomial $S(Z)$ will not have any roots outside the unit circle in the $Z$-plane. Consequently, the finite-difference scheme associated with $S(Z)$ will satisfy the von Neumann stability condition.

To find the stability conditions for a difference scheme, as a function of the parameters of interest, all the entries of the first column of the Routh table are forced to be nonnegative quantities. This leads to a set of algebraic inequalities that allow an analytical stability limit to be established without numerical root searching.

There is a special case for which there could still be roots in the right-half of the $r$-plane even if all of the entries of the first column of the Routh table are nonnegative. This case arises when all the elements in one row of the Routh table are zeros. For this special case, the Routh table can still be built up following the procedure given in [8].

### III. APPLICATION

To illustrate the application of the von Neumann method coupled with the R–H criterion to study the stability of the FDTD method, we have considered an extension of the conventional FDTD method that is able to incorporate dispersive media [9]. The stability of these schemes has been previously studied by using the von Neumann method and numerical root searching [2].

Consider the following time-dependent wave equation in a source-free homogeneous dispersive medium

$$\mu \frac{\partial^2 \vec{D}}{\partial t^2} - \nabla^2 \vec{E} = 0. \tag{4}$$

Before approximating this equation by using finite differences, several finite-difference operators are introduced. The central finite difference and the central average operators with respect to time are defined as

$$\delta_t F^n \equiv F^{n+(1/2)} - F^{n-(1/2)},$$
$$\mu_t F^n \equiv \left( F^{n+(1/2)} + F^{n-(1/2)} \right) \Big/ 2$$

respectively [10]. Note that these operators of second-order accuracy are defined over an interval $\Delta_t$. Analogous definitions apply with respect to the spatial coordinates.

According to Yee's scheme, (4) is approximated by

$$\mu \frac{\delta_t^2}{\Delta_t^2} \vec{D}^n - \sum_{\alpha=x,y,z} \frac{\delta_\alpha^2}{\Delta_\alpha^2} \vec{E}^n = 0 \tag{5}$$

where $\delta_\alpha$ ($\alpha = x, y, z$) denotes the central difference operator with respect to the coordinate indicated by the subscript. For the sake of simplicity, spatial indexes are not explicitly written since they are not necessary for the following development.

As mentioned in the preceding section, in order to apply the von Neumann method trial solutions (eigenfunctions) of the form, (1) should be substituted into (5), which usually requires quite tedious algebraic manipulations. A simpler alternative is used in this paper, which consists of replacing field quantities by their corresponding complex amplitudes

$$\vec{F} \to \vec{F}_0$$

and difference operators by their corresponding eigenvalues

$$\delta_\alpha \to \mathbf{j}2\sin\tilde{\theta}_\alpha, \qquad \alpha = x, y, z$$
$$\delta_t \to Z^{1/2} - Z^{-1/2}$$
$$\mu_t \to \left( Z^{1/2} + Z^{-1/2} \right) \Big/ 2$$

where $\tilde{\theta}_\alpha = \tilde{k}_\alpha \Delta_\alpha / 2$. Following this procedure, we have

$$(Z-1)^2 \vec{D}_0 + 4Z\epsilon_\infty \nu^2 \vec{E}_0 = 0 \tag{6}$$

where

$$\nu^2 = (c_\infty \Delta_t)^2 \sum_{\alpha=x,y,z} \frac{\sin^2 \tilde{\theta}_\alpha}{\Delta_\alpha^2} \tag{7}$$

and $c_\infty = 1/\sqrt{\mu\epsilon_\infty}$.

In addition to (4), a constitutive equation relating $\vec{D}$ and $\vec{E}$ must be considered. The differential form of this equation depends on the dielectric medium type, and its difference form depends on the discretization scheme. In the following, constitutive equations for Debye and Lorentz media along with the discretization schemes introduced in [9] will be considered.

### A. Debye Media

The constitutive equation for a first-order Debye medium can be expressed as

$$\left(\tau\epsilon_\infty \frac{d}{dt} + \epsilon_s\right)\vec{E} = \left(\tau\frac{d}{dt} + 1\right)\vec{D} \tag{8}$$

where $\tau$ is the relaxation time constant, and $\epsilon_s$ and $\epsilon_\infty$ are the static and the infinite-frequency permittivity, respectively.

Equation (8) is a first-order ordinary differential equation (ODE); thus, it can be discretized by simply using central finite differences. Consequently, the above equation in operational difference form reads

$$\left(\frac{\tau\epsilon_\infty}{\Delta_t}\boldsymbol{\delta}_t + \epsilon_s\boldsymbol{\mu}_t\right)\vec{E}^n = \left(\frac{\tau}{\Delta_t}\boldsymbol{\delta}_t + \boldsymbol{\mu}_t\right)\vec{D}^n. \tag{9}$$

As in the wave equation case, replacing the field quantities in (9) by their complex amplitudes, and operators by their eigenvalues, we have

$$\epsilon_\infty\left[(2\overline{\tau}+\overline{\epsilon}_s)Z + \overline{\epsilon}_s - 2\overline{\tau}\right]\vec{E}_0 = \left[(2\overline{\tau}+1)Z + 1 - 2\overline{\tau}\right]\vec{D}_0 \tag{10}$$

where $\overline{\epsilon}_s = \epsilon_s/\epsilon_\infty$ and $\overline{\tau} = \tau/\Delta_t$.

Equations (6) and (10) are a set of homogeneous linear equations. In order to seek nonzero solutions for the fields, the determinant of the coefficient matrix is equated to zero, leading to the stability polynomial

$$S_D(Z) = (\overline{\epsilon}_s + 2\overline{\tau})Z^3 + \left[4\nu^2(1+2\overline{\tau}) - \overline{\epsilon}_s - 6\overline{\tau}\right]Z^2$$
$$+ \left[4\nu^2(1-2\overline{\tau}) - \overline{\epsilon}_s + 6\overline{\tau}\right]Z + \overline{\epsilon}_s - 2\overline{\tau}. \tag{11}$$

Now, applying (3) to (11), we obtain the following stability polynomial in the $r$-plane

$$S_D(r) = \nu^2 r^3 + 2\overline{\tau}\nu^2 r^2 + \left(\overline{\epsilon}_s - \nu^2\right)r + 2\overline{\tau}\left(1-\nu^2\right).$$

The Routh table for this polynomial is

| | |
|---|---|
| $\nu^2$ | $\overline{\epsilon}_s - \nu^2$ |
| $2\overline{\tau}\nu^2$ | $2\overline{\tau}(1-\nu^2)$ |
| $\overline{\epsilon}_s - 1$ | $0$ |
| $2\overline{\tau}(1-\nu^2)$ | $0$ |

Forcing the entries of the first column of the above table to be nonnegative quantities, the following stability conditions are obtained:

$$0 \le \tau$$
$$\epsilon_\infty \le \epsilon_s$$
$$\nu^2 \le 1.$$

Taking (7) into account, the last inequality can be expressed as

$$\Delta_t \le \frac{1}{c_\infty}\left(\sum_{\alpha=x,y,z}\frac{\sin^2\tilde{\theta}_\alpha}{\Delta_\alpha^2}\right)^{-(1/2)}.$$

For practical calculations, the worst case is taken for $\sin^2\tilde{\theta}_\alpha$, i.e., $\sin^2\tilde{\theta}_\alpha = 1$. Thus, the above condition is reduced to

$$\Delta_t \le \frac{1}{c_\infty}\left(\sum_{\alpha=x,y,z}\frac{1}{\Delta_\alpha^2}\right)^{-(1/2)}. \tag{12}$$

Therefore, the difference scheme expressed by (5) and (9) preserves the stability limit of the conventional FDTD method.

### B. Lorentz Media

The constitutive equation for a Lorentz medium can be expressed as

$$\left(\epsilon_\infty\frac{d^2}{dt^2} + 2\delta_0\epsilon_\infty\frac{d}{dt} + \epsilon_s\omega_0^2\right)\vec{E} = \left(\frac{d^2}{dt^2} + 2\delta_0\frac{d}{dt} + \omega_0^2\right)\vec{D} \tag{13}$$

where $\omega_0$ is the resonant frequency and $\delta_0$ is the damping coefficient. This is a second-order ODE, thus, several centered schemes can be used for its finite-difference approximation. In operational form, the discretization given in [9] for this equation reads

$$0 = \left(\frac{\epsilon_\infty}{\Delta_t^2}\boldsymbol{\delta}_t^2 + \frac{2\delta_0\epsilon_\infty}{\Delta_t}\boldsymbol{\delta}_t\boldsymbol{\mu}_t + \epsilon_s\omega_0^2\boldsymbol{\mu}_{2t}\right)\vec{E}^n$$
$$- \left(\frac{1}{\Delta_t^2}\boldsymbol{\delta}_t^2 + \frac{2\delta_0}{\Delta_t}\boldsymbol{\delta}_t\boldsymbol{\mu}_t + \omega_0^2\boldsymbol{\mu}_{2t}\right)\vec{D}^n \tag{14}$$

where $\boldsymbol{\mu}_{2t}$ denotes the central average operator defined over a time interval $2\Delta_t$, i.e.,

$$\boldsymbol{\mu}_{2t}F^n = \left(F^{n+1} + F^{n-1}\right)/2.$$

Replacing the field quantities in (14) by their complex amplitudes and operators by their eigenvalues, the following equation is obtained:

$$\epsilon_\infty\left(\alpha_2 Z^2 - 4Z + \gamma_2\right)\vec{E}_0 - \left(\alpha_1 Z^2 - 4Z + \gamma_1\right)\vec{D}_0 = 0 \tag{15}$$

where

$$\alpha_1 = \overline{\omega}_0^2 + 2\overline{\delta}_0 + 2$$
$$\alpha_2 = \overline{\omega}_0^2\overline{\epsilon}_s + 2\overline{\delta}_0 + 2$$
$$\gamma_1 = \overline{\omega}_0^2 - 2\overline{\delta}_0 + 2$$
$$\gamma_2 = \overline{\omega}_0^2\overline{\epsilon}_s - 2\overline{\delta}_0 + 2$$

with $\overline{\omega}_0 = \omega_0\Delta_t$ and $\overline{\delta}_0 = \delta_0\Delta_t$.

Equations (6) and (15) are a set of homogeneous linear equations. Again, the stability polynomial is the determinant of the coefficient matrix

$$S_L(Z) = \left(\omega_0^2\overline{\epsilon}_s + 2\delta_0 + 2\right)Z^4$$
$$+ \left[4\nu^2\left(\overline{\omega}_0^2 + 2\overline{\delta}_0 + 2\right) - 2\overline{\omega}_0^2\overline{\epsilon}_s - 4\overline{\delta}_0 - 8\right]Z^3$$
$$+ 2\left(\overline{\omega}_0^2\overline{\epsilon}_s - 8\nu^2 + 6\right)Z^2$$
$$+ \left[4\nu^2\left(\overline{\omega}_0^2 - 2\overline{\delta}_0 + 2\right) - 2\omega_0^2\overline{\epsilon}_s + 4\overline{\delta}_0 - 8\right]Z$$
$$+ \overline{\omega}_0^2\overline{\epsilon}_s - 2\overline{\delta}_0 + 2. \tag{16}$$

After applying (3) to (16), the polynomial stability in the $r$-plane reads

$$S_L(r) = \sum_{i=0}^{4} b_i r^{4-i}$$

where

$$b_0 = \overline{\omega}_0^2\nu^2$$
$$b_1 = 4\overline{\delta}_0\nu^2$$
$$b_2 = \overline{\omega}_0^2\overline{\epsilon}_s + 4\nu^2$$
$$b_3 = 4\overline{\delta}_0\left(1-\nu^2\right)$$
$$b_4 = \overline{\omega}_0^2\left(\overline{\epsilon}_s - \nu^2\right) + 4\left(1-\nu^2\right).$$

For this case, the Routh table is

| $b_0$ | | $b_2$ | $b_4$ |
|---|---|---|---|
| $b_1$ | | $b_3$ | $0$ |
| $c_{3,0} = \dfrac{b_1 b_2 - b_0 b_3}{b_1}$ | | $c_{3,1} = b_4$ | $0$ |
| $c_{4,0} = \dfrac{c_{3,0} b_3 - b_1 b_4}{c_{3,0}}$ | | $0$ | $0$ |
| $c_{5,0} = b_4$ | | $0$ | $0$ |

According to the R–H criterion, we obtain the following stability conditions from the above table:

$$b_0 = \overline{\omega}_0^2 \nu^2 \geq 0$$
$$b_1 = 4\overline{\delta}_0 \nu^2 \geq 0$$
$$c_{3,0} = \overline{\omega}_0^2 \left(\overline{\epsilon}_s - 1 + \nu^2\right) + 4\nu^2 \geq 0$$
$$c_{3,0} b_3 - b_1 b_4 = 4\overline{\delta}_0 \overline{\omega}_0^2 \left(1 - 2\nu^2\right)(\overline{\epsilon}_s - 1) \geq 0$$
$$b_4 = \overline{\omega}_0^2 \left(\overline{\epsilon}_s - \nu^2\right) + 4\left(1 - \nu^2\right) \geq 0$$

which lead to

$$0 \leq \delta_0$$
$$\epsilon_\infty \leq \epsilon_s \qquad (17)$$

and

$$\nu^2 \leq 1/2. \qquad (18)$$

For practical calculations, the last condition is rewritten as

$$\Delta_t \leq \frac{1}{c_\infty \sqrt{2}} \left(\sum_{\alpha = x, y, z} \frac{1}{\Delta_\alpha^2}\right)^{-(1/2)}. \qquad (19)$$

Note that this stability condition is more restrictive than that for the conventional FDTD method.

For the sake of comparison with the stability condition given in [2] for the scheme considered in this section, (18) is expressed in one dimension as

$$\sin\tilde{\theta} \leq \frac{\Delta}{\sqrt{2} c_\infty \Delta_t}$$

which, for $c_\infty \Delta_t / \Delta = 1$, leads to

$$\sin\tilde{\theta} = \sin(\tilde{k}\Delta/2) \leq 1 \left/ \sqrt{2}\right.$$

thus

$$\tilde{k}\Delta \leq \pi/2.$$

This condition establishes the maximum value of the wavenumber such that an eigensolution (discrete plane wave) could propagate in a stable way. This condition agrees with that given in [2, Section II]. Note that in [2], this condition is associated with some values of the parameter $2\delta_0 \Delta_t$ (denoted by $h_1$ in [2]). However, according to (17), the only additional condition, found in this paper, on $\delta_0$ is that it must be nonnegative.

### C. New Difference Scheme for Lorentz Media

As mentioned in the preceding section, the stability condition given by (19) is more restrictive than that of the conventional FDTD method. This can be attributed to the fact that, while terms involving nonzeroth-order derivatives in (13) are approximated by difference operators defined over the interval $\Delta_t$, zeroth-order derivatives are approximated

by an operator defined over $2\Delta_t$. To overcome this limitation, we introduce the following difference scheme to approximate (13):

$$0 = \left(\frac{\epsilon_\infty}{\Delta_t^2} \boldsymbol{\delta}_t^2 + \frac{2\delta_0 \epsilon_\infty}{\Delta_t} \boldsymbol{\delta}_t \boldsymbol{\mu}_t + \epsilon_s \omega_0^2 \boldsymbol{\mu}_t^2\right) \vec{E}^n$$
$$- \left(\frac{1}{\Delta_t^2} \boldsymbol{\delta}_t^2 + \frac{2\delta_0}{\Delta_t} \boldsymbol{\delta}_t \boldsymbol{\mu}_t + \omega_0^2 \boldsymbol{\mu}_t^2\right) \vec{D}^n. \qquad (20)$$

All the terms are now approximated by operators defined over $\Delta_t$. It will be shown that this scheme preserves the stability limit of the conventional FDTD method.

Repeating the procedure carried out in the preceding section, the following stability polynomial in the $r$-plane is obtained for the present scheme:

$$S_{L'}(r) = \sum_{i=0}^{4} b_i r^{4-i}$$

where

$$b_0 = \nu^2 \overline{\omega}_0^2$$
$$b_1 = \nu^2 \overline{\delta}_0$$
$$b_2 = \overline{\omega}_0^2 \left(\overline{\epsilon}_s - \nu^2\right) + \nu^2$$
$$b_3 = \overline{\delta}_0 \left(1 - \nu^2\right)$$
$$b_4 = 1 - \nu^2.$$

According to the R–H criterion, the stability conditions for this scheme are

$$b_0 = \overline{\omega}_0^2 \nu^2 \geq 0$$
$$b_1 = \overline{\delta}_0 \nu^2 \geq 0$$
$$c_{3,0} = \nu^2 + \overline{\omega}_0^2 (\overline{\epsilon}_s - 1) \geq 0$$
$$c_{3,0} b_3 - b_1 b_4 = \overline{\omega}_0^2 \overline{\delta}_0 \left(1 - \nu^2\right)(\overline{\epsilon}_s - 1) \geq 0$$
$$b_4 = 1 - \nu^2 \geq 0$$

which leads to

$$0 \leq \delta_0$$
$$\epsilon_\infty \leq \epsilon_s$$

and

$$\nu^2 \leq 1.$$

Therefore, this new scheme preserves the stability limit of the conventional FDTD method.

### IV. CONCLUSION

In this paper, the combination of the von Neumann method with the R–H criterion has been used to derive closed-form stability expressions for the FDTD method—avoiding numerical root searching. This technique has been applied to determine the stability conditions of an extension of the FDTD method that is able to incorporate dispersive media previously reported in the literature. Both Debye and Lorentz media have been considered. It has been shown that, for the former case, the stability limit of the conventional FDTD method is preserved. However, for the latter case, a more restrictive stability limit was obtained. Therefore, a new scheme has been introduced to treat Lorentz media that recovers the stability limit of the conventional FDTD method.

### REFERENCES

[1] A. Taflove and M. E. Brodwin, "Numerical solution of steady-state electromagnetic scattering problems using the time-dependent Maxwell's equations," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 623–630, Aug. 1975.

[2] P. G. Petropoulos, "Stability and phase error analysis of FD-TD in dispersive dielectrics," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 62–69, Jan. 1994.

[3] S. A. Cummer, "An analysis of new and existing FDTD methods for isotropic cold plasma and a method for improving their accuracy," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 392–400, Mar. 1997.

[4] S. González, T. M. Hung-Bao, R. Gomez, and B. García, "On the application of finite methods in time domain to anisotropic dielectric waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 2195–2206, Dec. 1996.

[5] P. G. Petropoulos, "Analysis of exponential time-differencing for FDTD in lossy dielectrics," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 1054–1057, June 1997.

[6] J. A. Pereda, O. García, A. Vegas, and A. Prieto, "Numerical dispersion and stability analysis of the FDTD technique in lossy dielectrics," *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 45–47, July 1998.

[7] A. R. Mitchell and D. F. Griffiths, *The Finite Difference Method in Partial Differential Equations*. New York: Wiley, 1980.

[8] B. C. Kuo, *Automatic Control Systems*, 6th ed. Englewood Cliffs, NJ: Prentice-Hall, 1991.

[9] R. M. Joseph, S. C. Hagness, and A. Taflove, "Direct time integration of Maxwell's equations in linear dispersive media with absorption for scattering and propagation of femtosecond electromagnetic pulses," *Opt. Lett.*, vol. 16, no. 18, pp. 1412–1414, Sept. 1991.

[10] F. B. Hildebrand, *Introduction to Numerical Analysis*. New York: Mc-Graw-Hill, 1956.

# A Sub-Millimeter Accurate Microwave Multilevel Gauging System for Liquids in Tanks

Matthias Weiß and Reinhard Knöchel

*Abstract*—A microwave multilevel gauging system employing a frequency-stepped continuous-wave radar measurement technique is described in this paper. A conventional frequency-modulated continuous-wave radar technique is normally employed only to find the level of the liquid surface in storage tanks. The system described here also detects a second level, e.g., the tank floor or an impurity level. If this second reflection dominates, distance measurement with the inverse Fourier transform (IFT) results in poor resolution and shows a very high range error for small gaps between these two scatterers. For estimating the exact time delay and amplitude of the reflection from each scatterer, an optimal signal-processing algorithm is derived, based on a reference model. Performance of the multiple target-detection reference model is illustrated using measured data obtained with an HP-8510 network analyzer. It is demonstrated that the reference model offers a significant enhancement of resolution over the standard processing IFT algorithm and is insensitive to noise and clutter signals approach. The described system achieves a time-delay accuracy with a bandwidth of $\Delta f = 1$ GHz, which corresponds to a range error of $\pm 0.2$ mm.

*Index Terms*—FSCW radar, gauging, level measurement, microwave measuring, multilevel measurement, permittivity.

## I. INTRODUCTION

Measurement and control of liquid levels in storage tanks and processing vessels is important in many industrial processes. In storage tanks containing liquids with low transmission losses, e.g., mineral oil, reflection coefficient measurement shows a strong signal from the bottom of the tank. This unwanted dominant reflection in the radar channel increases the detectable minimum liquid level and raises the range error of the determined liquid surface. In these circumstances, one uses a gauging system with a sensor terminated by a matched load [1].

All industrial level gauging systems are limited to a finite bandwidth due an inexpensive production, which leads to an attractively price. The resolution offered by the inverse Fourier transform (IFT) for a narrow bandwidth is often unsatisfactory [2] and an alternative approach is required. Even if there exists no such restricted limitation in bandwidth, maximization of the achievable resolution for a particular measurement bandwidth is important.

In this paper, a microwave frequency-stepped continuous-wave (FSCW) radar system is described, which can monitor the liquid surface and the bottom of the tank or an impurity level within the tank simultaneously. To extract the time delays $\tau_1$ (liquid surface) and $\tau_2$ (bottom) accurately from the measured data, a multitarget reference model algorithm is used. This evaluation procedure is derived in the following section.

The benefits of the reference model over normally used evaluation algorithms, e.g., the IFT, is that deviations from an ideal scatterer like dispersion and windowing can be taken into account [3]. The accuracy of the determined range can then be made as that of an ideal target. In contrast to the decreased range resolution of the IFT for a windowed spectrum, the multitarget reference model resolves the delays of two adjacent scatterers with the same precision.

The resolution limitations of the IFT approach are demonstrated along with the enhancement offered by the reference model using physical measurements made with an HP-8510 network analyzer.

## II. DERIVATION OF THE REFERENCE MODEL

An FSCW system transmits a sequence of sinusoids at different frequencies and measures the steady-state amplitude and phase shift induced by the radar channel [4]. Fig. 1 shows a block diagram of such a radar system. A significant benefit of performing the measurements at discrete frequencies is that digital signal processing may be easily applied to the data. To maximize the range resolution achievable from an FSCW radar, a reference model technique is used. By this technique, a signal-processing computer produces a set of synthesized data at frequencies where the measurements were taken, and compares it to the physical measurements. The computerized data are based on a physical model of the transfer function of the radar channel [5], [6].

Finally, the algorithm has to minimize the difference between both the measured and synthesized data. After minimization is performed, the parameters of the reference model represent the ranging results. Best results were achieved with the least squares estimate, described by

$$FF = \sum_{k=k_0}^{k_0+N} [M_k - V_k] \cdot [M_k - V_k]^* \tag{1}$$

where the $M_k$ are the measured complex reflection coefficient pairs, $V_k$ are the values of the reference model, and $k$ is the index number of each measurement, which spans the integer range from $k_0$ to $k_0 + N$. $FF$ is the error function, which depends on the parameters of the reference model as yet not specified, and * denotes the complex conjugate.